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Existence results of Sobolev type neutral impulsive integrodifferential equations in Banach spaces

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Abstract

This paper deals with the study of existence of mild solutions of the neutral impulsive integrodifferential equations with finite delay in Banach spaces. The results are obtained via semigroup theory and we use the Schaefer's fixed point theorem to prove the main results. Finally an example is provided to prove the obtained the results.

Keywords: Existence, Integrodifferential equations, Neutral equations, Impulsive systems, Mild solution, Finite delay, Schafer fixed point theorem.

2000 Mathematics Subject Classification: 34G20, 34K30, 34K40, 47J35.

1 Introduction

In this paper, we discuss the existence of mild solutions of nonlinear impulsive neutral integrodifferential equations of Sobolev type of the form

$$\frac{d}{dt}\left[Bu(t) - g_1\left(t, u_t, \int_0^t h_1(t,s,u_s)ds\right)\right] = Au(t) + f(t,u_t) + g_2\left(t, u_t, \int_0^t h_2(t,s,u_s)ds\right),$$

$$t \in J = [0,a], \tag{1.1}$$

$$u_0 = \phi \text{ on } [-r,0], \tag{1.2}$$

$$\Delta u(t_i) = I_i(u_{t_i}),\ i = 1,2,\cdots,m_1, \tag{1.3}$$

where $0 \leq t_1 < t_2 < \cdots < t_p \leq a$, B and A are linear operators with domains contained in a Banach space X_1 and ranges contained in a Banach space Y. The functions $g_1, g_2 : J \times X \times X \to Y$, $h_1, h_2 : J \times J \times X \to X$, $f : J \times X \to Y$, $I_i : X \to Y$ are continuous functions, where $X = \{\psi : [-r,0] \to Y : \psi(t)$ is continuous everywhere except for a finite number of t_i points at which $\psi(t_i^+)$ and $\psi(t_i^-)$ exist and $\psi(t_i) = \psi(t_i^-)\}$. For any function $u \in \mathcal{PC}$ and for

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any $t \in J$, u_t denotes the function in X defined by $u_t(\theta) = u(t+\theta)$, $\theta \in [-r, 0]$ where $\mathcal{PC}$ defined later.

Neutral differential systems exist in different fields of applied mathematics and for this reason these systems have been investigated in the last few decades. Many applications with delayed argument exist in the derivatives of the state variable as well as in the independent variable, so it is called neutral differential equations. A neutral functional differential equation is one in which the derivatives of the ancient state or derivatives of functional of the ancient state are involved as well as the present state of the system. For more details about the existence of solution neutral functional differential equations, the readers can refer Hale and Verduyn Lunel [10] and the references [2, 6, 7, 9, 11].

Balachandran et al. [3–5], Dauer et al. [9] examined the existence of solutions of nonlinear neutral integrodifferential equations in Banach spaces. Hernandez et al. [11] acquired some existence results for abstract degenerate neutral functional differential equations. Annapoorani and Balachandran [1] analyzed the existence of solutions of partial neutral integrodifferential equations in Banach spaces. Using the Schaefer fixed point theorem Balachandran et al. [6] ascertained the existence results for nonlinear abstract neutral integrodifferential equation. The existence of mild solutions of neutral evolution integrodifferential equations has been discussed in [7].

Differential equations occur in many real world problems such as physics, population dynamics, ecology, biological systems, biotechnology, optimal control and so forth. Variety of work has been done with the assumption that the state variables and systems parameters change continuously. However, one may easily envision that hasty changes such as shock, harvesting and disasters may exist in nature. These phenomena are short time perturbations. Comparing to the duration of whole evolution process, its duration is negligible. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is in the form of impulses. The theory of impulsive differential equations [12, 16] is much wealthier than the corresponding theory of differential equations without impulsive effects. The impulsive condition

$$\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)), i = 1, 2, \cdots, m_1,$$

is a permutation of conventional initial value problems and short-term perturbations whose duration is negligible in association with the duration of the process. Lin and Liu [14] studied the iterative methods for the solution of impulsive functional differential systems. Sobolev type of equations arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics, and shear in second-order fluids. For more details, we refer the reader to [8, 13, 18].

Motivated by the above approach, and inspired by works in [19], the goal of this paper is to use the fixed point theorem to obtain the mild solutions of the nonlinear impulsive neutral integrodifferential equations of Sobolev type.

In Section 2, we establish some preliminary results of the nonlinear neutral impulsive integrodifferential equations of Sobolev type. In Section 3, we confer the existence of mild solutions by the Schaefer fixed point theorem. In the last section, an application is given to demonstrate the main results.

2 Preliminaries

In this section we recall some definitions, notations and results which are needed to establish our main results. Throughout this paper, $(X, \|\cdot\|)$ is a Banach space. We denote $C([0, a], X)$ the space of all X-valued functions on $[0,a]$ with norm $||x|| = \sup\{||x(t)|| : t \in [0,a]\}$. We also introduce the Banach space $\mathcal{PC}(J,R) = \{x : [-r,a] \to Y$ such that $x(\cdot)$ is continuous except for a finite numbe of pointst_i at which $x(t_i^+)$ and $x(t_i^-)$ exist and $x(t_i^-) = x(t_i)\}$ with the norm $||x||_{\mathcal{PC}} := \sup\{||x(t)|| : t \in [-r,a]\}$.

The operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the hypotheses:

($\mathbf{H_1}$) A and B are closed linear operators.

($\mathbf{H_2}$) $D(B) \subset D(A)$ and B is bijective.

($\mathbf{H_3}$) $B^{-1} : Y \to D(B)$ is continuous.

($\mathbf{H_4}$) The resolvent $R(\lambda, AB^{-1})$ is a compact operator for some $\lambda \in \rho(AB^{-1})$, the resolvent set of (AB^{-1})

The hypotheses ($\mathbf{H_1}$), ($\mathbf{H_2}$) and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \to Y$.

Lemma 2.1. *(See [15]) Let $T(t)$ be a strongly continuous semigroup and A be its infinitesimal generator. If the resolvent $R(\lambda : A)$ of A is compact for some $\lambda \in \rho(A)$ and $T(t)$is continuous in the uniform operator topology , then $T(t)$ is compact.*

From the above fact , AB^{-1} generates a compact semigroup $T(t),\ t > 0$ on Y.

Definition 2.2. *A solution $u : (-r,a) \to Y,\ a > 0$, is called a mild solution of the Cauchy problem* (1.1)-(1.3) *if*

(i) $u_0 = \phi$;

(ii) *the restriction of u(.) to the interval [0,a) is continuous;*

(iii) *for each $0 \le t < a$ the function $AB^{-1}T(t-s)g_1\left(s, u_s, \int_0^t h_1(s,\tau,u_\tau)d\tau\right)$, $s \in [0,t)$ is integrable and*

(iv) *the integral equation*

$$\begin{aligned} u(t) &= B^{-1}T(t)\left[B\phi - g_1(0,\phi,0)\right] + B^{-1}g_1\left(t, u_t, \int_0^t h_1(t,s,u_s)ds\right) \\ &+ \int_0^t B^{-1}AB^{-1}T(t-s)g_1\left(s, u_s, \int_0^t h_1(s,\tau,u_\tau)d\tau\right)ds + \int_0^t B^{-1}T(t-s)f(s,u_s)\,ds \\ &+ \int_0^t B^{-1}T(t-s)g_2\left(s, u_s, \int_0^t h_2(s,\tau,u_\tau)d\tau\right)ds + \sum_{0<t_i<t} B^{-1}T(t-t_i)I_i(u(t_i)), \quad t \in J, \end{aligned}$$

is satisfied.

Assume the following hypotheses hold:

($\mathbf{H_5}$) The strongly continuous semigroup of bounded linear operators $T(t)$ generates by A is compact and there exists a constant $M \ge 1$ such that $\|T(t)\| \le M$, for $t > 0$.

($\mathbf{H_6}$) There exist constants $a_1^*, a_2^* > 0$ and $a_3^* \ge 0$ with $a_1 = \max\{a_1^*, a_2^*\}$ such that $\|AB^{-1}T(t-s)g_1(t,\phi,v)\| \le a_1^*\|\phi\| + a_2^*|v| + a_3^*$, for all $t \in J,\ \phi \in C,\ v \in C$ and for $a_3 > 0$, $\|AB^{-1}T(t_1-s)g_1(s,\phi,v) - AB^{-1}T(t_2-s)g_1(s,\phi,v)\| \le a_3(|t_1-t_2|)$, for $t_1, t_2 \in J$.

(H_7) For each $s \in J, u \in C$, the function $h_1(\cdot, s, u) : J \to X$ is completely continuous, the function $h_1(\cdot, \cdot, u) : J \times J \to X$ is strongly measurable and $\{t \to h_1(t, s, u_s)\}$ is equicontinuous in $\mathcal{PC}([0, a], Y)$.

(H_8) For each $(t, s) \in J \times J$, the function $h_2(t, s, \cdot) : X \to x$ is continuous and, for each $u \in X$, $h_2(\cdot, \cdot, u) : J \times J \to X$ is strongly measurable.

(H_9) For each $t \in J$, the function $f(t, \cdot) : X \to Y$ is continuous and, for each $u \in X$, the function $f(\cdot, u) : J \to Y$ is strongly measurable.

(H_{10}) For each $t \in J$, the function $g_2(t, \cdot, \cdot) : X \times X \to Y$ is continuous and, for each $(u, v) \in X \times X$, the function $g_2(\cdot, u, v) : J \to Y$ is strongly measurable.

(H_{11}) There exist integrable functions $\alpha_i : J \to [0; \infty), i = 0, 1, 2$, such that

$$\|g_2(t, u, v)\| \leq \alpha_1(t)\Gamma_1(\|u\|) + \alpha_2(t)\Gamma_2(\|v\|), t \in J, \ u, v \in X,$$
$$\|f(t, u_t)\| \leq \alpha_0(t)\Gamma_0(\|u_t\|), \ 0 \leq t \leq a; \ u_t \in X,$$

where $\Gamma_i : [0, \infty) \to (0, \infty), i = 0, 1, 2$, are continuously differentiable nondecreasing functions, such that $\lim_{s\to\infty} \Gamma_0(s) = \infty, \ \Gamma_i', i = 0, 1, 2$, (the first derivative of Γ_i) are also nondecreasing and $\Gamma_0'(\|B\phi(0)\||B^{-1}|M) > 0$.

(H_{12}) The function $g_1 : J \times X \times X \to Y$ is completely continuous and, for any bounded set D in $\mathcal{PC}([-r, a], \ X)$, the set

$$\left\{t \to g_1\left(t, u_t, \int_0^t h_1(t, s, u_s)ds\right) : u \in D\right\}$$

is equicontinuous in $\mathcal{PC}([0, a], Y)$. There exist $\overline{a_1}, \overline{a_2} > 0$ and $\overline{a_3} \geq 0$ with $a_2 = \max\{\overline{a_1}, \overline{a_2}\}$ and $a_2 \in \left(0, \frac{1}{|B^{-1}|}\right)$such that $\|g_1(t, \phi, v)\| \leq \overline{a_1}\|\phi\| + \overline{a_2}\|v\| + \overline{a_3}$ for all $t \in J, \ \phi \in X, \ v \in X$. There exist $k_i : J \times J \to [0, \infty), \ i = 1, 2$, differentiable a.e., with respect to the first variable, such that $\int_0^t k_i(t, s)ds, \int_0^t \frac{\partial k_i}{\partial t}(t, s)ds$ are bounded on J and $\frac{\partial k_i}{\partial t}(t, s) \geq 0$ for a.e., $0 \leq s < t \leq a$. Moreover

$$|h_1(t, s, u)| \leq k_1(t, s)\theta_1(\|u\|), \ 0 \leq s < t \leq a, \ u \in X,$$
$$|h_2(t, s, u)| \leq k_2(t, s)\theta_2(\|u\|), \ 0 \leq s < t \leq a, \ u \in X,$$

where $\theta_i : [0, \infty) \to (0, \infty)$, i=1,2 are continuous nondecreasing functions.

(H_{13}) Let $p(t) = \max\left\{\alpha(t), \beta(t), \frac{M|B^{-1}|\gamma(t)}{1-|B^{-1}|a_2}, \frac{|B^{-1}|a_1}{1-|B^{-1}|a_2}\right\}$ such that

$$\int_0^a p(s)ds < \int_a^\infty \left\{\left[s + \theta_1(s) + \Gamma_0(s) + \Gamma_1(s) + \Gamma_2(L_0\theta_2(s))\right]\left[1 + \frac{\Gamma_1'(s)}{\Gamma_0'(s)}\right] + \frac{\theta_2(s)\Gamma_2'(L_0\theta_2(s))}{\Gamma_0'(s)}\right\}^{-1} ds,$$

where

$$\alpha(t) = \frac{|B^{-1}|}{1 - |B^{-1}|a_2}\left\{a_2k_1(t, t) + \int_0^t \left(a_2\frac{\partial k_1}{\partial t}(t, s) + a_1k_1(t, s)\right)ds\right\},$$
$$\beta(t) = k_2(t, t) + \int_0^t \left|\frac{\partial k_2}{\partial t}(t, s)\right|ds,$$
$$\gamma(t) = \max\{\alpha_0(t), \alpha_1(t), \alpha_2(t)\}.$$

L_0 is a finite bound for $\int_0^t k_2(t,s)ds$ and $b = \Gamma_0^{-1}\left(\Gamma_0(p_0) + \Gamma_1(p_0) + \Gamma_2(p_0)\right)$ with

$$p_0 = \frac{1}{1 - |B^{-1}|a_2}\left[\left(\|B\phi\| + \overline{a_1}\|\phi\| + \sum N_k\right)|B^{-1}|M + |B^{-1}|\overline{a_3}(1+M) + |B^{-1}|a_3^* a\right].$$

$(\mathbf{H_{14}})$ The maps $I_i : X \to Y$ are continuous and uniformly bounded. In the sequel ,we let $N_i = \sup\{\|I_i(u(t_i))\| : u \in X\}$.

Theorem 2.3. (Schaefer fixed point theorem)*(See [17]) Let X be a convex subset of a normed linear space V containing 0. If $F : X \to X$is a completely continuous operator, then either F has a fixed point or the subset $E(F) = \{y \in X : y = \lambda F y \ for \ some \lambda \in [0,1]\}$is unbounded*

We denote $R = \|B^{-1}\|$.

3 Existence results

This section deals with the existence of solutions for the problem (1.1)-(1.3).

Theorem 3.1. *Assume that the hypotheses* $(\mathbf{H_1})$-$(\mathbf{H_{14}})$ *hold. Then, the problem* (1.1)-(1.3) *admits a mild solution on* $[-r,a]$.

Proof. Consider the space $\mathcal{PC}_b = \mathcal{PC}([-r,a] : Y)$ endowed with the norm,

$$\|u\|_1 = \sup\{|u(t)| : -r \le t \le a\}$$

To prove the existence of mild solutions of (1.1)-(1.3), let us consider the nonlinear operator equation,

$$u(t) = \lambda N u(t), \ 0 < \lambda < 1$$

where $N : \mathcal{PC}_b \to \mathcal{PC}_b$ is given by

$$\begin{aligned} Nu(t) &= B^{-1}T(t)\big[B\phi - g_1(0,\phi,0)\big] + B^{-1}g_1\Big(t, u_t, \int_0^t h_1(t,s,u_s)ds\Big) \\ &+ \int_0^t B^{-1}AB^{-1}T(t-s)g_1\Big(s,u_s,\int_0^s h_1(s,\tau,u_\tau)d\tau\Big)ds + \int_0^t B^{-1}T(t-s)f(s,u_s)ds \\ &+ \int_0^t B^{-1}T(t-s)f\Big(s,u_s,\int_0^s h_2(s,\tau,u_\tau)d\tau\Big)ds + \sum_{0<t_i<t} B^{-1}T(t-t_i)I_i(u(t_i)), \quad t \in J. \end{aligned}$$

Now

$$\begin{aligned} |u(t)| &= |\lambda N u(t)| \\ &\le RM\big[\|B\phi\| + \overline{a_1}\|\phi\| + \overline{a_3}\big] + Ra_2\|u_t\| + R\overline{a_3} + Ra_2\int_0^t k_1(t,s)\theta_1(\|u_s\|)ds \\ &+ Ra_1\int_0^t \|u_s\|ds + Ra_1\int_0^t\Big(\int_0^s k_1(s,\tau)\theta_1(\|u_\tau\|)d\tau\Big)ds + Ra_3^* a \\ &+ RM\int_0^t \alpha_0(s)\Gamma_0(\|u_s\|)ds + RM\int_0^t \alpha_1(s)\Gamma_1(\|u_s\|)ds \\ &+ RM\int_0^t \alpha_2(s)\Gamma_2\Big(\int_0^s k_1(s,\tau)\theta_1(\|u_\tau\|)d\tau\Big)ds + RM\sum_{i=1}^{m_1} N_i. \end{aligned} \tag{3.1}$$

Let us define the function l by $l(t) = \sup\{|u(s)| : -r \le s \le t\}, t \in J$, then from (3.1) and our assumptions, we infer

$$\begin{aligned}
l(t) \le\ & RM\left[\|B\phi\| + \overline{a_1}\|\phi\| + \overline{a_3}\right] + Ra_2 l(t) + R\overline{a_3} + Ra_2\int_0^t k_1(t,s)\theta_1(l(s))ds \\
& + Ra_1\int_0^t l(s)ds + Ra_1\int_0^t\left(\int_0^s k_1(s,\tau)\theta_1(l(\tau))d\tau\right)ds + Ra_3^*a \\
& + RM\int_0^t \alpha_0(s)\Gamma_0(l(s))ds + RM\int_0^t \alpha_1(s)\Gamma_1(l(s))ds \\
& + RM\int_0^t \alpha_2(s)\Gamma_2\left(\int_0^s k_1(s,\tau)\theta_1(l(\tau))d\tau\right)ds + RM\sum_{i=1}^{m_1} N_i.
\end{aligned}$$

The above estimate is still valid, if $t^* \in [-r, 0]$, since $l(t) = \|\phi\|$ and $M \ge 1$.
Hence,

$$\begin{aligned}
l(t) \le\ & \frac{1}{1 - Ra_2}\Big\{RM\left[\|B\phi\| + \overline{a_1}\|\phi\| + \overline{a_3}\right] + R\overline{a_3} + Ra_2\int_0^t k_1(t,s)\theta_1(l(s))ds \\
& + Ra_1\int_0^t l(s)ds + Ra_1\int_0^t\left(\int_0^s k_1(s,\tau)\theta_1(l(\tau))d\tau\right)ds + Ra_3^*a \\
& + RM\int_0^t \alpha_0(s)\Gamma_0(l(s))ds + RM\int_0^t \alpha_1(s)\Gamma_1(l(s))ds \\
& + RM\int_0^t \alpha_2(s)\Gamma_2\left(\int_0^s k_1(s,\tau)\theta_1(l(\tau))d\tau\right)ds + RM\sum_{i=1}^{m_1} N_i\Big\}. \qquad (3.2)
\end{aligned}$$

Let us denote the right-hand side of (3.2) as $m(t)$. Then, clearly,

$$m(0) = \frac{1}{1 - Ra_2}\left\{RM\left(\|B\phi\| + \overline{a_1}\|\phi\| + \sum_{k=1}^{m} N_k\right) + R\overline{a_3}(1 + M) + Ra_3^*a\right\} \equiv p_0,$$

and $l(t) \le m(t), t \in J$. In addition, we have

$$\begin{aligned}
m'(t) = \frac{1}{1 - Ra_2}\Big\{ & Ra_2 k_1(t,t)\theta_1(l) + Ra_1 l(t) \\
& + Ra_1\int_0^t \frac{\partial k_1}{\partial t}(t,s)\theta_1(l(s))ds + Ra_1\int_0^t k_1(t,s)\theta_1(l)ds \\
& + RM\alpha_0(t)\Gamma_0(l(t)) + RM\alpha_1(t)\Gamma_1(l(t)) \\
& + RM\alpha_2(t)\Gamma_2\left(\int_0^t k_2(t,s)\theta_2(l(s))ds\right)\Big\} \ge 0,\ t \in J.
\end{aligned}$$

Next, let $n(t)$ be such that

$$\Gamma_0(n) = \Gamma_0(m) + \Gamma_1(m) + \Gamma_2\left(\int_0^t k_2(t,s)\theta_2(m)ds\right),$$

we have $m \leq n$, and by differentiation, and $(\mathbf{H_{12}})$, we get

$$\begin{aligned}
\Gamma_0'(n)n'(t) &= \Big(\Gamma_0'(m) + \Gamma_1'(m)\Big)m' + \Gamma_2'\Big(\int_0^t k_2(t,s)\theta_2(m)ds\Big) \\
&\quad \times \Big\{k_2(t,t)\theta_2(m) + \int_0^t \frac{\partial k_2}{\partial t}(t,s)\theta_2(m)ds\Big\} \\
&\leq \frac{\Gamma_0'(n) + \Gamma_1'(n)}{1 - Ra_2}\Big\{Ra_2 k_1(t,t)\theta_1(n) + Ra_1 n + Ra_2\int_0^t \frac{\partial k_1}{\partial t(t,s)}\theta_1(n)ds \\
&\quad + Ra_1\int_0^t k_1(t,s)\theta_1(n)ds + RM\gamma(t)\Big(\Gamma_0(n) + \Gamma_1(n) + \Gamma_2\Big(\int_0^t k_2(t,s)\theta_2(n)\Big)\Big)\Big\} \\
&\quad + \Big\{k_2(t,t) + \int_0^t \Big|\frac{\partial k_2}{\partial t}(t,s)\Big|ds\Big\}\theta_2(n)\Gamma_2'\Big(\theta_2(n)\int_0^t k_2(t,s)ds\Big). \qquad (3.3)
\end{aligned}$$

More over, by our assumptions on Γ_0', we have

$$\Gamma_0'(n) \geq \Gamma_0'(m) \geq \Gamma_0'(p_0) \geq \Gamma_0'(\|B\phi\|RM) > 0.$$

Therefore, (3.3)implies that

$$\begin{aligned}
n'(t) &\leq \Big(1 + \frac{\Gamma_1'(n)}{\Gamma_0'(n)}\Big)\Big\{\frac{Ra_1 n}{1 - Ra_2} \\
&\quad + \frac{R\theta_1(n)}{1 - Ra_2}\Big(a_2 k_1(t,t) + \int_o^t \Big(a_2\frac{\partial k_1}{\partial t}(t,s) + a_1 k_1(t,s)\Big)ds\Big) \\
&\quad + \frac{RM\gamma(t)}{1 - Ra_2}\Big(\Gamma_0(n) + \Gamma_1(n) + \Gamma_2\Big(\int_0^t k_2(t,s)\theta_2(n)ds\Big)\Big)\Big\} \\
&\quad + \frac{\theta_2(n)}{\Gamma_0'(n)}\Big(k_2(t,t) + \int_0^t \Big|\frac{\partial k_2}{\partial t}(t,s)\Big|ds\Big)\Gamma_2'\Big(\theta_2(n)\int_0^t k_2(t,s)ds\Big),
\end{aligned}$$

By $(\mathbf{H_{13}})$

$$\begin{aligned}
n'(t) &\leq \Big(1 + \frac{\Gamma_1'(n)}{\Gamma_0'(n)}\Big)\Big\{\frac{Ra_1 n}{1 - Ra_2} + \alpha(t)\theta_1(n) \\
&\quad + \frac{RM\gamma(t)}{1 - Ra_2}\Big(\Gamma_0(n) + \Gamma_1(n) + \Gamma_2\Big(L_0\theta_2(n)\Big)\Big)\Big\} \\
&\quad + \frac{\theta_2(n)\beta(t)}{\Gamma_0'(n)}\Gamma_2'\Big(L_0\theta_2(n)\Big) \\
&\leq p(t)\Big\{\Big[n + \theta_1(n) + \Gamma_0(n) + \Gamma_1(n) + \Gamma_2\Big(L_0\theta_2(n)\Big)\Big]\Big(1 + \frac{\Gamma_1'(n)}{\Gamma_0'(n)}\Big) \\
&\quad + \frac{\theta_2(n)}{\Gamma_0'(n)}\Gamma_2'\Big(L_0\theta_2(n)\Big)\Big\}.
\end{aligned}$$

Thus, for $0 \leq t \leq a$,

$$\int_{n(0)}^{n(t)} \left\{ \left[s+\theta_1(s)+\Gamma_0(s)+\Gamma_1(s)+\Gamma_2\Big(L_0\theta_2(s)\Big)\right]\left(1+\frac{\Gamma_1'(s)}{\Gamma_0'(s)}\right)+\frac{\theta_2(s)}{\Gamma_0'(s)}\Gamma_2'\Big(L_0\theta_2(s)\Big)\right\}^{-1} ds$$
$$\leq \int_0^a p(s)ds$$
$$< \int_a^\infty \left\{ \left[s+\theta_1(s)+\Gamma_0(s)+\Gamma_1(s)+\Gamma_2\Big(L_0\theta_2(s)\Big)\right]\left(1+\frac{\Gamma_1'(s)}{\Gamma_0'(s)}\right)+\frac{\theta_2(s)}{\Gamma_0'(s)}\Gamma_2'\Big(L_0\theta_2(s)\Big)\right\}^{-1} ds.$$

This implies that $n(t)$ must be bounded by some positive constant M_1 on $[0,a]$.

Consequently, $\|u\|_1 \leq M_1$. We shall now prove that the operator $N : \mathcal{PC}_b \to \mathcal{PC}_b$ defined by

$$Nu(t) = \phi(t), \ t \in [-r, 0]$$
$$\begin{aligned} Nu(t) &= B^{-1}T(t)\big[B\phi - g_1(0,\phi,0)\big] + B^{-1}g_1\Big(t, u_t, \int_0^t h_1(t,s,u_s)ds\Big) \\ &+ \int_0^t B^{-1}AB^{-1}T(t-s)g_1\Big(s,u_s,\int_0^t h_1(s,\tau,u_\tau)d\tau\Big) + \int_0^t B^{-1}T(t-s)f(s,u_s)ds \\ &+ \int_0^t B^{-1}T(t-s)g_2\Big(s,u_s,\int_0^t h_2(s,\tau,u_\tau)d\tau\Big)ds + \sum_{0<t_i<t} B^{-1}T(t-t_i)I_i(u(t_i)), \end{aligned}$$

is a completely continuous operator for all $t \in J$. Let $B_q = \{u \in \mathcal{PC}_b : \|u\| \leq q\}$ for some $q \geq 1$. We first show that N maps B_q into equicontinuous family. Let $u \in B_q$ and $t_1, t_2 \in [0,a]$. Then, if $0 \leq t_1 < t_2 \leq a$, (the other cases $t_1 < t_2 < 0$ and $t_1 < 0 < t_2$ may be treated similarly), we have

$$\begin{aligned} \|(Nu)(t_1)-(Nu)(t_2)\| \leq\ & R\|[T(t_1)-T(t_2)][B\phi - g_1(0,\phi,0)]\| \\ &+ R\Big\|g_1\Big(t_1,u_{t_1},\int_0^{t_1} h_1(t_1,s,u_s)ds\Big) - g_1\Big(t_2,u_{t_2},\int_0^{t_2} h_1(t_2,s,u_s)ds\Big)\Big\| \\ &+ R\int_0^{t_1}\Big\|AB^{-1}T(t_1-s)g_1\Big(s,u_s,\int_0^s h_1(s,\tau,u_\tau)d\tau\Big) \\ &- AB^{-1}T(t_2-s)g_1\Big(s,u_s,\int_0^s h_1(s,\tau,u_\tau)d\tau\Big)\Big\|ds \\ &+ R\int_{t_1}^{t_2}\Big\|AB^{-1}T(t_2-s)g_1\Big(s,u_s,\int_0^s h_1(s,\tau,u_\tau)d\tau\Big)\Big\|ds \\ &+ R\int_0^{t_1}\Big\|T(t_1-s)-T(t_2-s)\Big\|\|f(s,u_s)\|ds \\ &+ R\int_{t_1}^{t_2}\Big\|T(t_2-s)\Big\|\|f(s,u_s)\|ds \\ &+ R\int_0^{t_1}\Big\|T(t_1-s)-T(t_2-s)\Big\|\Big\|g_2\Big(s,u_s,\int_0^s h_2(s,\tau,u_\tau)d\tau\Big)\Big\|ds \\ &+ R\int_{t_1}^{t_2}\Big\|T(t_2-s)\Big\|\Big\|g_2\Big(s,u_s,\int_0^s h_2(s,\tau,u_\tau)d\tau\Big)\Big\|ds \\ &+ R\sum_{0<t_i<t}\|T(t_1-t_i)-T(t_2-t_i)\|N_i \end{aligned}$$

$$\begin{aligned}
&\le R\|[T(t_1)-T(t_2)][B\phi - g_1(0,\phi,0)]\| \\
&\quad + R\left\| g_1\left(t_1, u_{t_1}, \int_0^{t_1} h_1(t_1,s,u_s)ds\right) - g_1\left(t_2, u_{t_2}, \int_0^{t_2} h_1(t_2,s,u_s)ds\right)\right\| \\
&\quad + Ra_3\int_0^{t_1} |t_1-t_2|ds + R\int_{t_1}^{t_2} q_1(s)ds \\
&\quad + R\int_0^{t_1} \big\|T(t_1-s)-T(t_2-s)\big\| q_2(s)ds + R\int_{t_1}^{t_2} \big\|T(t_2-s)\big\| q_2(s)ds \\
&\quad + R\int_0^{t_1} \big\|T(t_1-s)-T(t_2-s)\big\| q_3(s)ds + R\int_{t_1}^{t_2} \big\|T(t_2-s)\big\| q_3(s)ds \\
&\quad + R\sum_{0<t_i<t} \big\|T(t_1-t_i)-T(t_2-t_i)\big\| N_i,
\end{aligned}$$

where

$$\begin{aligned}
q_1(s) &= a_1^*\|u_s\| + a_2^*\int_o^s k_1(s,\tau)\theta_1(\|u_\tau\|)d\tau + a_3^*, \\
q_2(s) &= \alpha_0(s)\Gamma_0(\|u_s\|), \\
q_3(s) &= \alpha_1(s)\Gamma_1(\|u_s\|) + \alpha_2(s)\Gamma_2\left(\int_0^s k_2(s,\tau)\theta_2(\|u_s\|)d\tau\right).
\end{aligned}$$

Since $T(t),\ t>0$, is compact and continuous in the uniform operator topology and by the assumptions on k_i,θ_i,Γ_i and the complete continuity of g_1, the right-hand side of the above inequality goes to zero as $(t_2-t_1)\to 0$. Therefore, the family $\{NB_q\}$ is equicontinuous. Moreover, for $u\ in\ B_q$,

$$\begin{aligned}
\|Nu(t)\| &\le |B^{-1}|\|T(t)\|\big|B\phi - g_1(0,\phi,0)\big| + |B^{-1}|q_1(t) + \int_0^t |B^{-1}|q_1(s)ds \\
&\quad + \int_0^t |B^{-1}|\|T(t-s)\|q_2(s)ds + \int_0^t |B^{-1}|\|T(t-s)\|q_3(s)ds \\
&\quad + \sum_{0<t_i<t} |B^{-1}|\|T(t-t_i)I_k(u(t_i))\| \\
&\le RM\big[\big\|B\phi\big\| + \overline{a_1}\|\phi\| + \overline{a_3}\big] + Rq_1(t) + R\int_0^t q_1(s)ds + RM\int_0^t q_2(s)ds \\
&\quad + RM\int_0^t q_3(s)ds + RM\sum_{0<t_i<t} N_i.
\end{aligned}$$

Hence NB_q is uniformly bounded, and consequently, according to Arzela-Ascoli's Theorem, it sufficient to show that NB_q is precompact in X. Next for fixed $t\in(0,a]$ and ϵ, such that $0<\epsilon<t$. We define, for $u\in B_q$,

$$\begin{aligned}
(N_\epsilon u)(t) &= B^{-1}T(t)\big[B\phi - g_1(0,\phi,0)\big] + B^{-1}g_1\left(t, u_t, \int_0^t h_1(t,s,u_s)ds\right) \\
&\quad + \int_0^{t-\epsilon} B^{-1}AB^{-1}T(t-s)g_1\left(s,u_s,\int_0^t h_1(s,\tau,u_\tau)d\tau\right)ds + \int_0^{t-\epsilon} B^{-1}T(t-s)f\big(s,u_s\big)ds \\
&\quad + \int_0^{t-\epsilon} B^{-1}T(t-s)g_2\left(s,u_s,\int_0^t h_2(s,\tau,u_\tau)d\tau\right)ds + RM\sum_{0<t_i<t} N_i.
\end{aligned}$$

As $T(t)$ is a compact operator, we see that $N_\epsilon B_q$ is precompact in X, for every ϵ, such that $0 < \epsilon < t$. On the other hand, we have

$$\begin{aligned}\|(Nu)(t) - (N_\epsilon u)(t)\| &\leq R\int_{t-\epsilon}^{t}\left\|AB^{-1}T(t-s)g_1\left(s, u_s, \int_0^t h_1(s,\tau,u_\tau)d\tau\right)\right\|ds \\ &+ RM\int_{t-\epsilon}^{t}\|f(s,u_s)ds\| + RM\int_{t-\epsilon}^{t}\left\|g_2\left(s,u_s,\int_0^t h_2(s,\tau,u_\tau)d\tau\right)\right\|ds \\ &\leq R\int_{t-\epsilon}^{t} q_1(s)ds + RM\int_{t-\epsilon}^{t} q_2(s)ds + RM\int_{t-\epsilon}^{t} q_3(s)ds.\end{aligned}$$

This shows that NB_q may be arbitrarily approached by precompact sets and hence it is a precompact subset of X.

Finally, we want to prove that $N : \mathcal{PC}_b \to \mathcal{PC}_b$ is continuous. Let $\{u_k(t)\}_{k=0}^{\infty} \subseteq \mathcal{PC}_b$ with $u_k \to u$ in $\mathcal{PC}_b$. Then there is an integer r such taht $\|u_k(t)\| \leq r$ for all k and $t \in J$. So $u_k \in B_r$. Moreover, by virtue of $(\mathbf{H_6})$, $(\mathbf{H_{11}})$ and $(\mathbf{H_{12}})$, we obtain

$$\begin{aligned}&\left\|g_1\left(t, u_{k_t}, \int_0^t h_1(t,s,u_{k_s})ds\right) - g_1\left(t,u_t,\int_0^t h_1(t,s,u_s)ds\right)\right\| \\ &\qquad \leq 2\left\{\overline{a_1}r + \overline{a_2}\left(\int_0^t k_1(t,s)\theta_1(r)ds\right) + \overline{a_3}\right\}, \\ &\|f(t,u_{k_t}) - f(t,u_t)\| \leq 2\alpha_0(t)\Gamma_0(r), \\ &\|g_2(t,u_{k_t},\int_0^t h_2(t,s,u_{k_s}))ds - g_2(t,u_t,\int_0^t h_2(t,s,u_s))\| \\ &\qquad \leq 2\{\alpha_1(t)\Gamma_1(r) + \alpha_2(t)\Gamma_2(L_0\theta_2(r))\}, \\ &\left\|AB^{-1}T(t-s)g_1\left(t,u_{k_t},\int_0^t h_1(t,s,u_{k_s})ds\right) - AB^{-1}T(t-s)g_1\left(t,u_t,\int_0^t h_1(t,s,u_s)ds\right)\right\|ds \\ &\qquad \leq 2\left\{a_1^* r + a_2^*\left(\int_0^t k_1(t,s)\theta_1(r)ds\right) + a_3^*\right\}.\end{aligned}$$

From assumptions $(\mathbf{H_6})$-$(\mathbf{H_{10}})$ and $(\mathbf{H_{12}})$

$$\begin{aligned}&g_1\left(t,u_{k_t},\int_0^t h_1(t,s,u_{k_s})ds\right) \to g_1\left(t,u_t,\int_0^t h_1(t,s,u_s)ds\right), \\ &f(t,u_{k_t}) \to f(t,u_t), \\ &g_2\left(t,u_{k_t},\int_0^t h_2(t,s,u_{k_s})\right)ds \to g_2\left(t,u_t,\int_0^t h_2(t,s,u_s)\right), \\ &AB^{-1}T(t-s)g_1\left(t,u_{k_t},\int_0^t h_1(t,s,u_{k_s})ds\right) \to AB^{-1}T(t-s)g_1\left(t,u_t,\int_0^t h_1(t,s,u_s)ds\right),\end{aligned}$$

as $k \to \infty$, for each $t \in J$. Also, we have

$$\|Nu_k - Nu\| \leq \sup_{t\in J}\left\{\left\|B^{-1}\left[g_1\left(t,u_{k_t},\int_0^t h_1(t,s,u_{k_s})ds\right) - g_1\left(t,u_t,\int_0^t h_1(t,s,u_s)ds\right)\right]\right\|\right.$$

$$
\begin{aligned}
&+\Big\|\int_0^t B^{-1}\Big[AB^{-1}T(t-s)g_1\Big(s,u_{k_s},\int_0^s h_1(s,\tau,u_{k_\tau})d\tau\Big)\\
&-AB^{-1}T(t-s)g_1\Big(s,u_s,\int_0^s h_1(s,\tau,u_\tau)d\tau\Big)\Big]ds\Big\|\\
&+\Big\|\int_0^t B^{-1}T(t-s)[f(s,u_{k_s})-f(s,u_s)]\Big\|\\
&+\Big\|\int_0^t B^{-1}T(t-s)\Big[g_2\Big(s,u_{k_s},\int_0^s h_2(s,\tau,u_{k_\tau})d\tau\Big)\\
&-g_2\Big(s,u_s,\int_0^s h_2(s,\tau,u_\tau)d\tau\Big)\Big]ds\Big\|\\
&+\sum_{0<t_i<t}\Big\|B^{-1}T(t-t_i)\Big[I_i(u_k(t_i))-I_i(u(t_i))\Big]\Big\|\Big\}\\
\leq\ & R\Big\|g_1\Big(t,u_{k_t},\int_0^t h_1(t,s,u_{k_s})ds\Big)-g_1\Big(t,u_t,\int_0^t h_1(t,s,u_s)ds\Big)\Big\|\\
&+R\int_0^a\Big\|\Big[AB^{-1}T(t-s)g_1\Big(s,u_{k_s},\int_0^s h_1(s,\tau,u_{k_\tau})d\tau\Big)\\
&-AB^{-1}T(t-s)g_1\Big(s,u_s,\int_0^s h_1(s,\tau,u_\tau)d\tau\Big)\Big]ds\Big\|\\
&+RM\int_0^a\Big\|f(s,u_{k_s})-f(s,u_s)\Big\|+RM\int_0^a\Big\|g_2\Big(s,u_{k_s},\int_0^s h_2(s,\tau,u_{k_\tau})d\tau\Big)\\
&-g_2\Big(s,u_s,\int_0^s h_2(s,\tau,u_\tau)d\tau\Big)ds\Big\|+RM\sum_{0<t_i<t}\Big\|I_i(u_k(t_i))-I_k(u(t_i))\Big\|\\
\to\ & 0 \text{ as } \quad k\to\infty.
\end{aligned}
$$

Thus from Dominated Convergence theorem , N is continuous and consequently, N is completely continuous operator.

Finally the set $\xi(N)=\{u\in\mathcal{PC}:u=\lambda Nu,\ \lambda\in(0,1)\}$ is bounded. Consequently by Schaefer's theorem the operator N has a fixed point in $\mathcal{PC}_b$. Thus the problem (1.1)-(1.3) has at least one mild solution on $[-r,a]$. □

4 An example

Consider the partial integrodifferential equations of neutral type

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big[z(t,x)-z_{xx}(t,x)+\int_{-\infty}^t a_1(s-t)z_t(s,x)ds\Big]\\
&=z_{xx}(t,x)+\rho(t,z(t,x))+f\left(t,z(x,t-r),\int_0^t a(t,s,z(x,s-r))ds\right),\quad x\in[0,\pi],\ t\in I,\\
&z(t,0)=z(t,\pi)=0,\quad t\in I,\\
&z(0,x)=\phi(x,t)\quad -r\leq t\leq 0,\\
&\Delta z|_{t=t_i}=I_i(z(x))=(\gamma_i(z(x))+t_i)^{-1},\quad z\in X,\ 1\leq i\leq p, \qquad (4.1)
\end{aligned}
$$

where ϕ is continuous, a,a_1,f are continuous functions and satisfy certain smoothness conditions.

Take $X = Y = L^2[0;\pi]$ and the constant γ_i is small. Let us take

$$\begin{aligned}
&g_1\left(t, x_t, \int_0^t h_1(t,s,x_s)ds\right) = -\int_{-\infty}^t a_1(s-t)z_t(s,x)ds,\\
&f(t,x_t) = \rho(t, z(t,x)),\\
&g_2\left(t, x_t, \int_0^t h_2(t,s,x_s)ds\right) = f\left(t, z(x,t-r), \int_0^t h(t,s,z(x,s-r))ds\right),\\
&I_i(z(x)) = (\gamma_i(z(x)) + t_i)^{-1}.
\end{aligned}$$

Define the operator $A : \mathcal{D}(A) \subset X \to Y$ and $B : \mathcal{D}(B) \subset X \to Y$ by

$$Az = -z_{xx}, \quad Bz = z - z_{xx},$$

where each domain $\mathcal{D}(A)$ and $\mathcal{D}(B)$ is given by

$$\{z \in X : z, z_x \text{ are absolutely continuous, } z_{xx} \in X, z(0) = z(\pi) = 0\}.$$

Then the above problem can be formulated abstractly as

$$\begin{aligned}
&\frac{d}{dt}\left[Bu(t) - g_1\left(t, x_t, \int_0^t h_1(t,s,x_s)ds\right)\right] = Au(t) + f(t,x_t) + g_2\left(t, x_t, \int_0^t h_2(t,s,x_s)ds\right),\\
&\qquad\qquad t \in (0,a],\ t \neq t_k,\\
&u(0) = \phi,\\
&\Delta u(t_k) = I_k(x_{t_k}), \quad k = 1,2,\ldots,m, \qquad (4.2)
\end{aligned}$$

Then A and B can be written, respectively, as

$$Az = \sum_{n=1}^{\infty} n^2\langle z, z_n\rangle z_n,\ z \in \mathcal{D}(A),$$
$$Bz = \sum_{n=1}^{\infty} (1+n^2)\langle z, z_n\rangle z_n,\ z \in \mathcal{D}(B),$$

where $z_n(x) = \sqrt{2/\pi}\sin(nx)$, $n = 1,2,\ldots$, is the orthogonal set of vectors of A. Furthermore for $z \in X$, we have

$$B^{-1}z = \sum_{n=1}^{\infty} \frac{1}{1+n^2}\langle z, z_n\rangle z_n,$$
$$-AB^{-1}z = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}\langle z, z_n\rangle z_n,$$
$$T(t)z = \sum_{n=1}^{\infty} \exp\left(\frac{-n^2 t}{1+n^2}\right)\langle z, z_n\rangle z_n.$$

It is easy to see that AB^{-1} generates a strongly continuous semigroup $T(t)$ on Y and $T(t)$ is compact such that $\|T(t)\| \le M$ and $\|AB^{-1}g(t,\phi,y)\| \le M_1$ for each $t > 0$.

The function $\rho : J \times [0,\pi] \to [0,\pi]$ is completely continuous and there exists a constant $n_1 > 0$ such that

$$\|\rho(t, z(t,x))\| \le n_1.$$

Also, the functions $h : J \times J \times [0, \pi] \to [0, \pi]$ and $f : J \times [0, \pi] \times [0, \pi] \to [0, \pi]$ are measurable and there exist integrable functions $l_1, l_2 : J \to [0, \infty)$ and $l_3 : J \times J \to [0, \infty)$ such that

$$\|f(t, x, y)\| \leq l_1(t)\Gamma_1(\|x\|) + l_2(t)\Gamma_2(\|y\|),$$
$$|h(t, s, x)| \leq l_3(t, s)\Phi_1(\|x\|),$$

where $\Gamma_i : [0, \infty) \to (0, \infty), i = 1, 2.$ are continuously differentiable nondecreasing functions, $\Phi_1 : [0, \infty) \to (0, \infty)$ is continuously nondecreasing function.

Let $p(t) = \max\{b(t), |B^{-1}|Mq(t)\}$ be such that

$$\int_0^a p(s)ds < \int_a^\infty \left\{[\Gamma_1(s) + \Gamma_2(L_0\Phi_1(s))]\frac{\Gamma_1'(s)}{\Gamma_0'(s)} + \frac{\Phi_1(s)}{\Gamma_0'(s)}\Gamma_2'(s)(L_0\Phi_1(s))\right\}^{-1} ds,$$

where

$$b(t) = \left\{l_3(t, t) + \int_0^t \left|\frac{\partial l_3(t, s)}{\partial t}\right| ds\right\},$$
$$q(t) = \max\{l_1(t), l_2(t)\}.$$

L_0 is a finite bound for $\int_0^t l_3(t, s)ds$ and $b = \Gamma_0^{-1}(\Gamma_1(\alpha_0) + \Gamma_2(0))$ with
$\alpha_0 = |B^{-1}|M [\|B\phi\| + n_1] + |B^{-1}|n_1 + |B^{-1}M_1 a.$
Further, all the conditions stated in the Theorem 3.1 are satisfied. Hence the equation (4.1) has a mild solution on $[0, \pi]$.

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